Also, it was shown recently by Omar de la Cruz that Δ_3-finite and III-finite are independent, neither implies the other in ZF. In $\mathcal{N}3$, Mostowski's linearly ordered model, $\mathcal{P}(A)$, where A is the set of atoms, is Dedekind finite, but can be linearly ordered. Therefore, $\mathcal{P}(A)$ is III-finite, but not Δ_3-finite.

On the other hand, in Fraenkel's Basic Model, $\mathcal{N}1$, the set of all finite subsets of A, is III-infinite, but it contains no infinite linearly ordered subset. For suppose that L is such an infinite linearly ordered subset and E is a support for L and its linear order R. Then, there exists $n \in \omega$ such that $L' = \{x \in L : |x| = n\}$ is infinite; otherwise we can well order L by

$$x < y \text{ iff } |x| < |y| \text{ or } (|x| = |y| \text{ and } xRy).$$

Then we can define a map from A onto ω, which is impossible.

Now, since there are only finitely many subsets of E, there exist distinct $u, v \in L'$ such that

$$u \cap E = v \cap E.$$

It is easy to find a permutation σ that fixes E and such that $\sigma(u) = v$, $\sigma(v) = u$. This contradicts the assumption that R is a linear order supported by E.

It is also easy to get a permutaion model which is half Mostowski and half basic Fraenkel, in which there exist sets which are Δ_3-finite and not III-finite and other sets which are III-finite, but not Δ_3 finite.

Finally, the sentence:

"$(\exists A, B)$ (A is III-finite and A is not Δ_3-finite, and B is Δ_3-finite and B is not III-finite)"

is boundable, therefore the result transfers to ZF.

This leaves several problems open. For example, what is the relationship between Δ_5 and V$''$?