Additions to Part IV: Notes

Eliminate 6 and 10 as "unused" notes.

I. New note 6.

NOTE 6

Definitions of Compact. There are several definitions of "compact" for topological spaces which are equivalent in ZFC, but not in ZF^0. In this note we describe what is known about the relationships between these definitions in ZF^0. (Also, see note 10.)

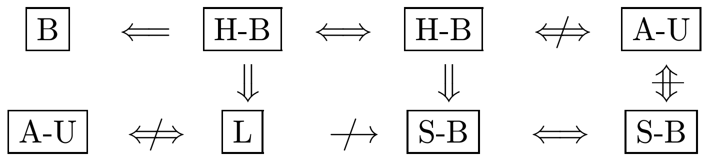
Definition. Assume (X, T) is a topological space.

1. (X, T) is *Heine-Borel compact* (or H-B compact or compact) if every open cover has a finite subcover. (It is shown in Herrlich [1996a] that (X, T) is H-B compact if and only if every filter has an accumulation point.)
2. (X, T) is *Alexandroff-Urysohn compact* (or A-U compact) if every infinite subset E of X has a complete accumulation point, (i.e,, a point $x_o \in X$ such that for every neighborhood U of x_0, $|E \cap U| = |E|$).
3. (X, T) is *subbase compact* (or S-B compact) if there is a subbase S for T such that every open cover by elements of S has a finite subcover.
4. (X, T) is *Bourbaki compact* (or B compact) if every ultrafilter converges in X. (An (ultra)filter $\mathcal{F}$ *converges* to a point x in X if evey neighborhood of x is in $\mathcal{F}$.)
5. (X, T) is *linearly compact* (or L compact) if every nest of non-empty closed sets has a non-empty intersection.
6. (X, T) is *sequentially compact* (or SEQ compact) if every sequence has a convergent subsequence.
7. (X, T) is *Weierstrauss compact* (or W compact) if every infinite subset has an accumulation point.

In Howard [1990] it is shown that in $\mathcal{N}1$ the set of atoms with the discrete topology is linearly compact but not subbase compact. It is also shown that in $\mathcal{N}56$, the topological space X described in Howard [1990] is A-U compact but not S-B compact. (The model $\mathcal{N}56$ is $\mathcal{M}2$ in Howard [1990].)

The following argument that X is not L-compact in $\mathcal{N}56$ is due to Adrienne Stanley. (Refer to Howard [1990] for definitions.) Let $g : \omega \to [0, 1] \cap \mathbb{Q}$ be a bijection. Define by induction $U_0 = \bigcup\{B_{\{r\}} : r \in \mathbb{Q} \setminus [0, 1]\}$ and $U_{n+1} = U_n \cup B_{g(n)}$. Then $\{U_n : n \in \omega\}$ is a nest of open sets that covers X but has no finite subcover.

In Howard [1990] and Herrlich [1996a] it is shown that, for a given topological space X the implications in the following diagram hold. (In the diagram "H-B" means "Heine-Borel compact", "A-U" means "Alexandroff-Urysohn compact", etc., and "$A \to B$" means " A implies B", "$A \not\to B$" means "A does not imply B", "$A \Rightarrow B$" means " A implies B, but B does not imply A", "$A \not\Leftrightarrow B$" means "A does not imply B and B does not imply A", and "$A \Leftrightarrow B$" means "A implies B and B implies A".)



Finally we mention a result from Herrlich [1996a]: The assertion that products of B compact spaces are B compact is equivalent to the statement "AC (form 1) or there does not exist a non-principal ultrafilter (the negation of form 206)".

II. New Note 10.

NOTE 10

Standard definitions from point set topology. In this note we give standard definitions from point set topology which do not occur elsewhere.

Definition.

1. A *pseudometric* on X is a function $d : X \times X \to X$ such that
   - $\forall x, y \in X$, $d(x, y) \geq 0$
   - $d(x, y) + d(y, z) \geq d(x, z)$
   - $d(x, x) = 0$.

   In the following (X, d) is a pseudometric space.
2, $S(y, r)$ is the ball of radius r centered at y.
3. (X, d) is *complete* if every Cauchy net converges. (In note 28, a metric space is defined to be complete or Frechet complete if Cauchy sequences converge. When this definition of complete is intended we will refer to note 28.)
4. (X, d) is *totally bounded* (or TBd) if for every positive real number r there is a finite set Y such that $X = \bigcup_{y \in Y} S(y, r)$.
5. (X, d) is *sequentially bounded* if every sequence in X has a Cauchy subsequence.
6. $(x_\lambda)_{\lambda \in \Lambda}$ is a *Cauchy net* in (X, d) if $(\forall r \in \mathbb{R})(\exists \lambda_0 \in \Lambda)(\forall \lambda_1, \lambda_2 \in \Lambda)$(If $\lambda_1, \lambda_2 \geq \lambda_0$, then $\lambda_1, \lambda_2 \in S(x, r)$ for some $x \in X$).
7. $\mathcal{F}$ is a *Cauchy filter* in (X, d) if $(\forall r \in \mathbb{R})(\exists x \in X)(S(x, r) \in \mathcal{F})$.

The diagram below from Bentley/Herrlich [1998] gives some relationships in ZF^0 between various forms of compactness for pseudometric spaces. (In the diagram "H-B" means " Heine-Borel compact", "A-U" means "Alexandroff-Urysohn compact", "C-TBd" means "complete and totally bounded, "SEQ" means "sequentially compact", "W" means "Weierstrass compact" (see note 6), "$A \Rightarrow B$" means "A implies B, but B does not imply A", "$A \not\Leftrightarrow B$" means "A does not imply B and B does not imply A", and "$A \Leftrightarrow B$" means "A implies B and B implies A".)

$$\begin{array}{ccc}
\boxed{\text{A-U}} & \not\Leftrightarrow & \boxed{\text{H-B}} \\
\Downarrow & & \Updownarrow \\
\boxed{\text{W}} & \Leftarrow & \boxed{\text{H-B}} \\
\Downarrow & & \Downarrow \\
\boxed{\text{SEQ}} & \Leftarrow & \boxed{\text{C-TBd}}
\end{array}$$

Definition. Assume (X, T) is a topological space.

1. A *zero-set* in X is a set of the form $\{x \in X : f(x) = 0\}$ where $f : X \to [0, 1]$ is continuous.
2. A *closed* (respectively *open, clopen, z-*) *filter* is a filter in the lattice of all closed (respectively open, clopen, zero-) sets in X.
3. A filter $\mathcal{F}$ in a lattice L is *prime* if for all A and B in L, if $A \cup B \in \mathcal{F}$ then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.
4. A subset $\mathcal{B}$ of a filter in $\mathcal{P}(X)$ is a *filter base* for a filter $\mathcal{F}$ if $\forall B_1, B_2 \in \mathcal{B}$, there is a $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$ and $\mathcal{F} = \{C : B \subseteq C \text{ for some } B \in \mathcal{B}\}$.
5. A *free filter* is a non-principal filter.

Definition. Let F be a family of functions from X to Y where X and Y are topological spaces. The *compact-open topology on* F is the topology with subbase consisting of all sets of the form $\{f \in F : f[K] \subseteq U\}$ where K is a compact subset of X and U is an open subset of Y.

Definition. A set F of functions from X to Y, where Y is a metric space is *equicontinuous* if $\forall x \in X$, $\forall \epsilon > 0$, $\exists$ a neighborhood U of x such that $\forall f \in F$ and $\forall y \in U$, $d(f(x), f(y)) < \epsilon$.

III. Add to note 120:

59. $8 + 384 \leftrightarrow 1$. Herrlich/Steprans [1997]
60. $10 + 388 \leftrightarrow 43$. Keremedis [1999].
61. $30 + 392 \leftrightarrow 1$, clear.
62. $30 + 395 \leftrightarrow 67$, clear.
63. $30 + 400 \leftrightarrow 15$, clear.

IV. Change the equation in note 104 to:

$$(*) \quad \begin{aligned}
&(\forall E \in S)[A - E \neq \emptyset \\
&\qquad \wedge (\forall a \in A - E)(\forall b \in A - \{a\})(\exists \sigma \in \mathcal{G})[((\forall c \in E)(\sigma(c) = c) \\
&\qquad \wedge (\sigma(a) \neq a \wedge \sigma(b) = b)) \\
&\qquad \vee (\sigma(a) = b \wedge \sigma(b) = a \wedge (\forall c \in A - \{a, b\})\sigma(c) = c)]].
\end{aligned}$$

V. In note 28 rewrite item 8 as follows:

8. (X, T) is *Baire* or is a *Baire space* if the intersection of each countable sequence of dense, open sets in X is dense in X. (Equivalently, (X, T) is Baire if and only if X is not the union of a countable sequence of nowhere dense sets.)

VI. In note 18, add forms 139 and 389 to the list of forms that are transferable and true in every FM model.

VII. New note 149.

Add the following to the table of contents of the notes:

NOTE 149

We give a proof that [14 CY] implies 14. (Abian [1973] proves that form 99 (Rado's lemma) + form 62 ($C(\infty, < \aleph_0)$) implies [14 CY] and 99 + 62 is form [14 Y].) Assume [14 CY] and let S be a collection of finite, non-empty sets. Choose an element t_0 in $\bigcup S$ and define an operation $*$ on $\bigcup S$ by $t * s = t_0$ for all t and s in $\bigcup S$. (Any binary operation defined on $\bigcup S$ would be good enough for our purpose.) For each $a \in S$ let x_a be a variable whose domain is $D_a = a$. Let $\mathcal{G}$ be the system consisting of all equations of the form $x_{a_1} * x_{a_2} * \cdots * x_{a_n} = t_0$ Where $a_1, \ldots, a_n$ are distinct elements of S. The system $\mathcal{G}$ satisfies the hypotheses of 390 and therefore $\mathcal{G}$ has a partial solution $(x_a = d_a)_{a \in S}$. Since for every $a \in S$, $d_a \in D_a = a$, we have a choice function for S.