Additions to Part IV: Notes

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New note:

## NOTE 151

In this note we give the definition of a lattice ordered group and related notions for forms [14 DA]– [14 DF].

**Definition.**

1. If $(G, +)$ is a group and $\leq$ is a partial ordering on $G$ compatible with $+$ such that each pair of elements in $G$ have a least upper bound and a greatest lower bound, then $(G, +)$ is called a *lattice ordered group* or an *$l$-group*. (The partial order $\leq$ is *compatible* with $+$ if for all $a, b, c$, and $d$ in $G$, if $a \leq b$, and $c \leq d$, then $a + c \leq b + d$.
2. If $(G, +)$ is an $l$-group, $G$ is called *archimedian* if for all pairs of strictly positive elements $g, h \in G$, there is an $n \in \mathbb{N}$ such that $g^n \leq h$ does not hold. (An element $g \in G$ is *positive* if $e \leq g$ and *strictly positive* if $e \leq g$, where $e$ is the unit.)
3. If $(G, +)$ is an $l$-group, $G$ is called *hyperarchimedian* if all its homomorphic images are archimedian.
4. An $l$-group is called *simple* if it does not contain any proper $l$-ideals.
5. Two elements $g$ and $h$ in an $l$-group $G$ are called *orthogonal* if the meet of their absolute values is $e$, the unit. (The *absolute value* of an element $g \in G$ is $g$ if $e \leq g$ and $-g$ otherwise, where $-g$ is the inverse of $g$.)
6. A *weak unit* in an $l$-group is a positive element $u$ which is orthogonal only to $e$.
7. A topological space is called *Boolean* if it is compact, Hausdorf, and the family of compact open subsets form a base for the topology.
8. $A$ is called a *Boolean product* of the family $\mathcal{A} = \{A_i : i \in I\}$ if
   (i) $A$ is a subdirect product of $\mathcal{A}$.
   (ii) $I$ admits a Boolean space topology such that
   (a) For any atomic formula $\phi(v_1, v_2, \cdots, v_n)$ and for all $a_1, a_2, \cdots, a_n \in A$, the subset

$$\{i \in I : A_i \models \phi(a_1(i), a_2(i), \cdots, a_n(i)\}$$

   is clopen in $I$.
   (b) For all $a, b \in A$ and for all clopen $J \subseteq I$, the element $(a \restriction J) \cup (b \restriction (I \setminus J))$ belongs to $A$.

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# NOTE 152

A proof that form 409 implies form 62. Let $X$ be a set of disjoint finite sets and construct the graph $(G, \Gamma)$ where $G = \bigcup X$ and for $t \in G$, if $t \in y \in X$ then $\Gamma(t) = y \setminus \{t\}$. (So if $y \in X$ there is an edge from every element of $y$ to every other element of $y$.) Let $K = \{0, 1\}$ and define the function $T : \mathcal{P}(K) \to \mathcal{P}(K)$ by $T(\{0\}) = \{1\}$, $T(\{1\}) = \{0\}$, $T(\emptyset) = K$ and $T(K) = K$. The hypotheses of 409 are satisfied for $(G, \Gamma)$ and $K$. Here is the argument:

If $A$ is a finite subgraph of $G$ (I.e., $A \subseteq G$), the $A$ can be written as the disjoint union $A = (A \cap y_1) \cup \cdots (A \cap y_n)$ where $y_i \in X$ and $y_i \cap A \neq \emptyset$, for $i = 1, \ldots, n$. Choose one element $t_i$ in each of the sets $A \cap y_i$ for $i = 1, \ldots n$ and define $\phi_A : A \to K$ by $\phi_A(t_i) = 0$ for $i = 1, \ldots, n$ and $\phi_A(t) = 1$ for other elements $t \in A$. For the proof that $\phi_A(t) \in T(\phi_A(\Gamma_A(t)))$ for all $t \in A$, ($\Gamma_A$ is $\Gamma$ restricted to $A$.) assume that $t \in A \cap y_i$ and consider two cases

Case 1. $|A \cap y_1| = 1$. In this case $t = t_i$ so $\phi_A(t) = \phi_A(t_i) = 0$ Also in this case $\Gamma_A(t) = \emptyset$ so $T(\phi_A(\Gamma_A(t))) = T(\phi_A(\emptyset)) = T(\emptyset) = K = \{0, 1\}$ so $\phi_A(t) \in T(\phi_A(\Gamma_A(t)))$.

Case 2. $|A \cap y_i| > 1$. In this case let $s$ be an element of $A \cap y_i$ different from $t_i$. If $t = t_i$ then $s \in \Gamma_A(t_i) = \Gamma_A(t)$ so $\phi_A(s) \in \phi_A(\Gamma_A(t))$ so $1 \in \phi_A(\Gamma_A(t))$. By the definition of $T$ it follows that $0 \in T(\phi_A(\Gamma_A(t)))$. But $\phi_A(t) = \phi_A(t_i) = 0$. On the other hand if $t \neq t_i$ then $t_i \in \Gamma_A(t)$ so $\phi_A(t_i) = 0 \in \phi_A(\Gamma_A(t))$. Using the definition of $T$ again, this means that $1 \in T(\phi_A(\Gamma_A(t)))$. But $\phi_A(t) = 1$.

Therefore by 409, there is a function $\phi_A : G \to K$ such that for all $t \in G$, $\phi(t) \in T(\phi(\Gamma(t)))$. We now argue that for each $y \in X$, $\phi$ restricted to $y$ cannot be constant. (By contradiction.) Assume $\forall t \in y$, $\phi(t) = 0$. Then for any $t \in y$, $T(\phi(\Gamma(t))) = T(\{0\}) = \{1\}$. But then $\phi(t) = 0 \notin T(\phi(\Gamma(t)))$. A similar argument shows that $\phi(t) = 1$ for all $t \in y$ is impossible. We can now get a Kinna-Wagner function for $X$ by defining $f(y) = \{t \in y : \phi(t) = 0\}$ for each $y \in X$.