Additions to Part IV: Notes

Change the first part of note 32 to the following:

NOTE 32

Notes for forms [159 A] and [159 B] from Blass [1983b].

A *similarity type* τ is a set J of operation symbols and for each $j \in J$, an arity I_j (I_j indexes the argument places of j). An *algebra of type* τ is a set A and for each $j \in J$, an operation j_A from A^{I_j} into A. The following is a theorem of ZF. (See Blass [1983b] and Kerkhoff [1965] [1969]. Also see notes 50 and 115.)

Theorem. *For any similarity type τ and set V there is an algebra $F(V)$ of type τ (called the τ (or olutely free algebra generated by V) with the property that $V \subseteq F(V)$ and for any A of type τ and any $\eta : V \to A$ there is a unique homomorphism $\alpha : F(V) \to A$ with $\alpha(v) = v$ for all $v \in V$.*

Add the following as the last sentence of the first paragraph of note 50.
(Also see notes 32 and 115.)

Change the last sentence of note 115 to:
See also forms [1 AP], [1 AQ], [1 AR], [67 E] and [67 F] and notes 32 and 50.

In the table of contents for the notes add:

New note:

NOTE 151

In this note we give the definition of a lattice ordered group and related notions for forms [14 DA]– [14 DF].

Definition.

1. If $(G, +)$ is a group and $\leq$ is a partial ordering on G compatible with $+$ such that each pair of elements in G have a least upper bound and a greatest lower bound, then $(G, +)$ is called a *lattice ordered group* or an *l-group*. (The partial order $\leq$ is *compatible* with $+$ if for all a, b, c, and d in G, if $a \leq b$, and $c \leq d$, then $a + c \leq b + d$.
2. If $(G, +)$ is an l-group, G is called *archimedian* if for all pairs of strictly positive elements $g, h \in G$, there is an $n \in \mathbb{N}$ such that $g^n \leq h$ does not hold. (An element $g \in G$ is *positive* if $e \leq g$ and *strictly positive* if $e \leq g$, where e is the unit.)
3. If $(G, +)$ is an l-group, G is called *hyperarchimedian* if all its homomorphic images are archimedian.
4. An l-group is called *simple* if it does not contain any proper l-ideals.

5. Two elements g and h in an l-group G are called *orthogonal* if the meet of their absolute values is e, the unit. (The *absolute value* of an element $g \in G$ is g if $e \leq g$ and $-g$ otherwise, where $-g$ is the inverse of g.)
6. A *weak unit* in an l-group is a positive element u which is orthogonal only to e.
7. A topological space is called *Boolean* if it is compact, Hausdorf, and the family of compact open subsets form a base for the topology.
8. A is called a *Boolean product* of the family $\mathcal{A} = \{A_i : i \in I\}$ if
   (i) A is a subdirect product of $\mathcal{A}$.
   (ii) I admits a Boolean space topology such that
   (a) For any atomic formula $\phi(v_1, v_2, \cdots, v_n)$ and for all $a_1, a_2, \cdots, a_n \in A$, the subset

$$\{i \in I : A_i \models \phi(a_1(i), a_2(i), \cdots, a_n(i)\}$$

   is clopen in I.
   (b) For all $a, b \in A$ and for all clopen $J \subseteq I$, the element $(a \restriction J) \cup (b \restriction (I \setminus J)$ belongs to A.

In the table of contents for the notes add:

NOTE 152

A proof that form 409 implies form 62. Let X be a set of disjoint finite sets and construct the graph (G, Γ) where $G = \bigcup X$ and for $t \in G$, if $t \in y \in X$ then $\Gamma(t) = y \setminus \{t\}$. (So if $y \in X$ there is an edge from every element of y to every other element of y.) Let $K = \{0, 1\}$ and define the function $T : \mathcal{P}(K) \to \mathcal{P}(K)$ by $T(\{0\}) = \{1\}$, $T(\{1\}) = \{0\}$, $T(\emptyset) = K$ and $T(K) = K$. The hypotheses of 409 are satisfied for (G, Γ) and K. Here is the argument:

If A is a finite subgraph of G (I.e., $A \subseteq G$), the A can be written as the disjoint union $A = (A \cap y_1) \cup \cdots (A \cap y_n)$ where $y_i \in X$ and $y_i \cap A \neq \emptyset$, for $i = 1, \ldots, n$. Choose one element t_i in each of the sets $A \cap y_i$ for $i = 1, \ldots n$ and define $\phi_A : A \to K$ by $\phi_A(t_i) = 0$ for $i = 1, \ldots, n$ and $\phi_A(t) = 1$ for other elements $t \in A$. For the proof that $\phi_A(t) \in T(\phi_A(\Gamma_A(t)))$ for all $t \in A$, (Γ_A is Γ restricted to A.) assume that $t \in A \cap y_i$ and consider two cases

Case 1. $|A \cap y_1| = 1$. In this case $t = t_i$ so $\phi_A(t) = \phi_A(t_i) = 0$ Also in this case $\Gamma_A(t) = \emptyset$ so $T(\phi_A(\Gamma_A(t))) = T(\phi_A(\emptyset)) = T(\emptyset) = K = \{0, 1\}$ so $\phi_A(t) \in T(\phi_A(\Gamma_A(t)))$.

Case 2. $|A \cap y_i| > 1$. In this case let s be an element of $A \cap y_i$ different from t_i. If $t = t_i$ then $s \in \Gamma_A(t_i) = \Gamma_A(t)$ so $\phi_A(s) \in \phi_A(\Gamma_A(t))$ so $1 \in \phi_A(\Gamma_A(t))$. By the definition of T it follows that $0 \in T(\phi_A(\Gamma_A(t)))$. But $\phi_A(t) = \phi_A(t_i) = 0$. On the other hand if $t \neq t_i$ then $t_i \in \Gamma_A(t)$ so $\phi_A(t_i) = 0 \in \phi_A(\Gamma_A(t))$. Using the definition of T again, this means that $1 \in T(\phi_A(\Gamma_A(t)))$. But $\phi_A(t) = 1$.

Therefore by 409, there is a function $\phi_A : G \to K$ such that for all $t \in G$, $\phi(t) \in T(\phi(\Gamma(t)))$. We now argue that for each $y \in X$, ϕ restricted to y cannot be constant. (By contradiction.) Assume $\forall t \in y$, $\phi(t) = 0$. Then for any $t \in y$, $T(\phi(\Gamma(t))) = T(\{0\}) = \{1\}$. But then $\phi(t) = 0 \notin T(\phi(\Gamma(t)))$. A similar argument shows that $\phi(t) = 1$ for all $t \in y$ is impossible. We can now get a Kinna-Wagner function for X by defining $f(y) = \{t \in y : \phi(t) = 0\}$ for each $y \in X$.

Change the beginning of note 32 to the following:

NOTE 32

Notes for forms [159 A] and [159 B] from Blass [1983b].

A *similarity type* τ is a set J of operation symbols and for each $j \in J$, an arity I_j (I_j indexes the argument places of j). An *algebra of type* τ is a set A and for each $j \in J$, an operation j_A from A^{I_j} into A. The following is a theorem of ZF. (See Blass [1983b] and Kerkhoff [1965] [1969]. Also see notes 50 and 115.)

Theorem. *For any similarity type τ and set V there is an algebra $F(V)$ of type τ (called the τ (or absolutely free algebra generated by V) with the property that $V \subseteq F(V)$ and for any A of type τ and any $\eta : V \to A$ there is a unique homomorphism $\alpha : F(V) \to A$ with $\alpha(v) = v$ for all $v \in V$.*

At the end of the first paragraph of note 50 add:
(Also see notes 32 and 115.)

Change the last sentence of note 115 to:
Also see forms [1 AP], [1 AQ], [1 AR], [67 E] and [67 F] and notes 32 and 50.

In the table of contents for the notes change the description for note 23 to the following:

Revised note 23:

NOTE 23

Definitions for forms [14 Q], [52 E], [52 N], and 410-412.

If E is a normed linear space or a Banach space (complete, normed, linear space) E^* (the *dual* of E) is the space of all bounded (continuous) linear functionals on E. (That is, all $f : E \to \mathbb{R}$ such that $(\exists M > 0)(\forall x \in E)(|f(x)| \le \|M\|)$.) E^* is a Banach space if we define $\|f\| = \sup_{x \ne 0} \left(\frac{|f(x)|}{\|x\|} \right)$. Each $x \in E$ can be thought of as a linear functional on E^* if we define $x(f) = f(x)$ for all f in E^*. The *weak* topology* on E^* is the weakest topology that makes all these linear functionals continuous. (A sequence $\{f_n\}$ in E^* is said to be *weakly* convergent* if $\lim_{n\to\infty} f_n(x)$ exists for every $x \in E$. A sequence $\{x_n\}$ in E is said to be *weakly convergent* if $\lim_{n\to\infty} f(x_n)$ exists for every $f \in E^*$.) We let E^{**}

be the dual of E^*. If $\phi : E \to E^{**}$ such that for each $x \in E$, $\phi(x) = f_x$, where for all $g \in E^*$, $f_x(g) = g(x)$, then ϕ is called the *natural embedding* of E into E^{**}. If the natural embedding is onto, the space is called *reflexive*.

Sets of the form $w_{x,\epsilon} = \{f \in E^* : |f(x)| < \epsilon\}$ form a basis for the weak* topology. If $X \subseteq E$, X is *convex-compact* if whenever F_i, for $i \in I$, are closed convex subsets of X and $\{X \cap F_i : i \in I\}$ has the finite intersection property, then $\bigcap_{i\in I}(X \cap F_i) \neq \emptyset$. Note that the definition of *convex-compact* may be given in any topological vector space. E is said to be *uniformly convex* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = \|y\| = 1$, $\|x - y\| \geq \epsilon$ implies $\frac{1}{2}\|x + y\| \leq 1 - \delta$. (A *topological vector space (linear topological space)* is a vector space (linear space) with a topology in which the operations of addition and scalar multiplication are continuous. An *affine subspace*, A, of a topological vector space E is a translation of a subspace of E, $A = v + S = \{v + w : w \in S\}$, where S is a subspace of E and $v \in E$.)

(The main reason that form [14 Q] implies 410 is that if the space E is reflexive, then there is a mapping of E onto E^{**} and the weak topology on E corresponds to the weak* on E^{**}.)

medskip Add to the end of note 10

Definition. If $(f_n)_{n\in\omega}$ is a sequence of continuous functions from a topological space X to a topological space Y and $f : X \to Y$, then $(f_n)_{n\in\omega}$ *converges continuously* to f provided $\forall x \in X$, and for all sequences $(x_n)_{n\in\omega}$ of elements of X such that $(x_n)_{n\in\omega} \to x$, $(f(x_n))_{n\in\omega} \to f(x)$.

Add to the table of contents of the notes:

Add note 153:

NOTE 153

Definitions for Determinateness Axioms (form [94 R]).

Definition. Assume that A is a set of sequences of of elements from the set X.

1. The game $G(X^{\mathbb{N}}, A)$ is the game played by two players I and II who alternately choose elements of X (with I choosing first) to produce a sequence $x = (x_o, x_1, x_2, \ldots)$. Player I wins if $x \in A$, otherwise II wins.
2. A *strategy* for the game $G(X^{\mathbb{N}}, A)$ is a function from finite sequences of elements of X to X.
3. If s is a strategy for the game $G(X^{\mathbb{N}}, A)$ and $x \in X^{\mathbb{N}}$, then $s[x]$ is the sequence $(s(\emptyset), x_0, s(x_0), x_1, s(x_0, x_1), x_2, s(x_0, x_1, x_2), \ldots)$ and $[x]s$ is the sequence $(x_0, s(x_0), x_1, s(x_0, x_1), x_2, s($ ($s[x]$ is the sequence obtained when player I plays the strategy s and player II plays the sequence x. Similarly for $[x]s$.)
4. If s is a strategy for the game $G(X^{\mathbb{N}}, A)$, then s is a *winning strategy for player I* if for all $x \in X^{\mathbb{N}}$, $s[x] \in A$. s is a *winning strategy for player II* if for all $x \in X^{\mathbb{N}}$, $[x]s \notin A$.