Additions to Part IV: Notes

Add the following as a new paragraph at the end of note 40:

The following are definitions for [94 S]. Suppose \( x \in \mathbb{R} \) and \( X \subseteq \mathbb{R} \).

(a) \( x \) is called an accumulation point of \( X \) if every open neighborhood of \( x \) contains a point in \( X - \{x\} \).

(b) \( x \) is called a cluster point of \( X \) if every open neighborhood of \( x \) contains an infinite number of points in \( X \).

(c) \( x \) is called a limit point of \( X \) if there is a sequence \( \{x_n : n \in \omega\} \subseteq X \) such that for every open neighborhood \( N_x \) of \( x \), there is real number \( M > 0 \) such that for all \( n > M, x_n \in N_x \).

In any \( T_1 \) space, \( (\mathbb{R} \text{ with the order topology is } T_1) \), a point is a cluster point if and only if it is an accumulation point. (Clearly, a cluster point is an accumulation point. Suppose \( x \) is an accumulation point of \( X \) that is not a cluster point. Suppose \( N_x \) is a neighborhood of \( x \) that only contains a finite number of elements of \( X, x_1, x_2, \ldots, x_n \). Since \( X \) is \( T_1 \), using induction, we can find a neighborhood \( M \) of \( x \) that does not intersect \( \{x_1, x_2, \ldots, x_n\} \). Then \( N_x \cap M \) is a neighborhood of \( x \) that does not contain any points of \( X \) different from \( x \). This contradicts the fact that \( x \) is an accumulation point.)