Additions to Part IV: Notes, January 2001

New Notes

NOTE 154

Definitions from constructive order theory (for forms [0 AT], [0 AU], [1 DH], [1 DI], [1 DJ], [67 AB], [67 AC], [144 B] through [144 M] and 413 through 416).

Definition. A *subset selection* $\mathcal{Z}$ is a (class) function defined on the class of all partial orderings such that if $(P, \leq)$ is a partial ordering then $\mathcal{Z}(P, \leq)$ (or $\mathcal{Z}P$ for short) is a subset of $\mathcal{P}(P)$. The subset selection $\mathcal{Z}$ is a *subset system* if for any two partial orderings $(P_1, \leq_1)$ and $(P_2, \leq_2)$ if $f : P_1 \to P_2$ is order preserving then for all $Z \in \mathcal{Z}P_1$, $f[Z] \in \mathcal{Z}P_2$.

In Erné [2000] the following subset selectors are considered

a. $\mathcal{A}P = \mathcal{P}(P)$ (arbitrary subsets of P)
b. $\mathcal{B}P =$ one or two element subsets of P
c. $\mathcal{C}P =$ all non-empty chains in $(P, \leq)$
d. $\mathcal{D}P =$ all directed subsets in $(P, \leq)$ ($Z \subseteq P$ is *directed* by $\leq$ if any two elements of Z have a common upper bound.)
e. $\mathcal{E}P =$ all one element subsets of P
f. $\mathcal{F}P =$ all finite subsets of P
g. $\mathcal{W}P =$ all non-empty subsets of P well ordered by $\leq$.
h. $\mathcal{U}P =$ all $U \subseteq P$ such that $\forall x, y \in U$, $x \vee y$ exists in P and is in $U =$ all $\bigvee$-subsemilattices of P.

Note that all of the above except $\mathcal{U}$ are subset systems.

Definition. Assume $(P \leq)$ is a partially ordered set and $\mathcal{Z}$ is a subset selection.

1. $(P, \leq)$ is *$\mathcal{Z}$-complete* (or $\mathcal{Z}$-$\bigvee$-complete) if every element Z of $\mathcal{Z}$ has a sup (denoted $\bigvee Z$ or $\bigvee_P Z$).
2. A subset X of P is *$\mathcal{Z}$-subcomplete* if each $Z \in \mathcal{Z} \cap \mathcal{P}(X)$ has a supremum s in P that is contained in X
3. If $Y \subset P$ then s is a *constructive supremum* of Y if either $Y = \emptyset$ and s is the least element of P (if it exists) or s is an upper bound of Y and there is a function $\psi : P \to Y$ such that $(\forall x \in P)(s \not\leq x \leftrightarrow \psi(x) \not\leq x)$.
4. $(P, \leq)$ is *constructively $\mathcal{Z}$-complete* if each $Z \in \mathcal{Z}P$ has a constructive supremum.
5. A *constructively complete lattice* is a complete lattice in which every supremum is constructive.
6. $\mathcal{Z}^\vee$ is the class function whose domain is the class of all partial orderings and which is defined by

$$\mathcal{Z}^\vee P = \{Y \subseteq P : (\forall x \in Y)(\forall z \in P)(z \leq x \to z \in Y) \text{ and } (\forall Z \in \mathcal{P}(Y) \cap \mathcal{Z}P)(\text{If } x = \bigvee_P Y \text{ then} x \in Y\}$$

7. $(P, \leq)$ is *constructively directed if there is a function assigning an upper bound to each non-empty finite subset of* P.
8. *If* S *is a set and* $\mathcal{X} \subseteq \mathcal{P}(S)$ *then* $\mathcal{X}$ *is* $\mathcal{Z}$*-inductive (or* $\mathcal{Z}$*-union complete if for each* $\mathcal{Y} \in \mathcal{Z}\mathcal{X}$, $\bigcup \mathcal{Y} \in \mathcal{X}$.
9. ${}^{c}\mathcal{D}P$ *(or* ${}^{c}\mathcal{D}(P, \leq)$*) is the system of all constructively directed subsets of* P.
10. *If* S *is a set and* $\mathcal{X} \subseteq \mathcal{P}(S)$ *then* $\mathcal{X}$ *is a closure system (on* S*) if* $\mathcal{X}$ *is closed under arbitrary intersections (with* $\bigcap \emptyset = S$*).*
11. *An element* x *of a* $\mathcal{Z}$*-complete poset* P *is* $\mathcal{Z}$*-compact if for all* $Z \in \mathcal{Z}P$, *if* $x \leq \bigvee Z$, *then* $x \in \downarrow Z$. *(*$\downarrow Z = \{t \in P : (\exists y \in Z)(t \leq y)\}$. *The* $\mathcal{D}$*-compact elements are called the compact elements.*
12. *A* $\mathcal{Z}$*-complete poset is* $\mathcal{Z}$*-compactly generated if each of its elements is a supremum of* $\mathcal{Z}$*-compact elements.*
13. *An algebraic lattice is a* $\mathcal{D}$*-compactly generated complete lattice.*
14. *If* S *is a set and* $\mathcal{X} \subseteq \mathcal{P}(S)$ *then* $\mathcal{X}$ *is a system of finite character for all* X, $X \in \mathcal{X} \Leftrightarrow \mathcal{F}X \subseteq \mathcal{X}$
15. *A Scott closed subset of* P *is an element of* $\mathcal{D}^{\vee}P$.
16. *A map* $\phi : P \to Q$ *between posets preserves* $\mathcal{Z}$*-suprema if for each* $Z \in \mathcal{Z}P$ *having a supremum* s, *the image* $\phi(s)$ *is the supremum of* $\phi[Z]$.
17. *A* $\mathcal{Z}$*-frame is a complete lattice in which the distributive law* $x \wedge \bigvee Z = \bigvee(x \wedge Z)$ *holds for all* $x \in P$ *and all* $Z \in \mathcal{Z}P$.

Note 155

A proof that 144 is true in $\mathcal{N}14$, $\mathcal{N}15$, $\mathcal{N}17$, $\mathcal{N}18$, $\mathcal{N}36(\beta)$, $\mathcal{N}37$, *and* $\mathcal{N}41$. *We will prove that form [144 B], the set induction principle, is true in* $\mathcal{N}41$. *The proof can be modified to show that 413 is true in the other models listed above. It suffices to show that in* $\mathcal{N}41$, *every set* S *is the union of a family* Y *of well ordered sets such that* Y *is well ordered by* $\subseteq$. *Assume* S *is a set in* $\mathcal{N}41$. *For each* $m \in \omega$, *let* $S_m = \{z \in S : z$ *has a support contained in* $\bigcup_{n \leq m} B_n\}$. *(See the description of* $\mathcal{N}41$ *for the definition of* B_n.*) Each element of* S_m *has support* $\bigcup_{n \leq m} B_n$ *and therefore* S_m *is well ordered in* $\mathcal{N}41$. *Further,* S_m *has empty support and therefore the ordering* $\leq *$ *on* $Y = \{S_m : m \in \omega\}$ *defined by* $S_m \leq *S_k$ *if and only if* $m \leq k$ *is in the model. But this ordering has order type* ω. *Since it is clear that* $S = \bigcup Y$ *the proof is completed.*

Note 156

A proof that 144 and [144 B] are equivalent. For the proof we need an generalization of the function W *defined in note 25. For any class* C *we define the function* W_C *on ordinals as follows:* $W_C(0)$ *is the set of well orderable subsets of* C, $W_{\alpha+1} = \{\bigcup Q : Q \subseteq W_\alpha$ *and* Q *is well orderable*$\}$ *and (for limit ordinals* λ*)* $W_C(\lambda) = \bigcup_{\beta<\lambda} W_C(\beta)$. *(Then for each ordinal* α, $W(\alpha) = W_V(\alpha)$ *where* V *is the universe.) Proceeding with the proof we first assume that form 144 (the set induction principle) is true and we let* S *be any set. We need to show that* S *is almost well orderable, i.e., that* $S \in \bigcup_{\alpha \in On} W(\alpha)$. *Since* *it is clear from the definition that* $\bigcup_{\alpha\in On} W_S(\alpha) \subseteq \bigcup_{\alpha\in On} W(\alpha)$*, it suffices to show that* $S \in \bigcup_{\alpha\in On} W_S(\alpha)$*. But this follows from the fact that* $\bigcup_{\alpha\in On} W_S(\alpha)$ *satisfies the hypotheses of the set induction principle.*

Now assume that that every set is almost well orderable (form 144). Let X *and* S *satisfy the hypotheses of the set induction principle. By form 144,* $S \in W(\alpha)$ *for some ordinal* α*. Using the definition of* $W_S(\alpha)$ *and induction on ordinals we obtain* $W_S\beta = W(\beta) \cap \mathcal{P}(S)$ *for all ordinals* β*. Therefore* $S \in W_S(\alpha)$*. Another easy induction argument using the fact that* X *and* S *satisfy the hypotheses of the set induction principle gives us* $W_S(\beta) \subseteq X$ *for all ordinals* β*. Hence we can conclude that* $S \in X$*.*

Since it is known that form 144 does not imply the axiom of multiple choice (form 67), this answers a question of Erné [2000]*: Does the set induction principle imply the axiom of multiple choice?*

Add to note 120:

66. $10 + 144 \to 9$, *Erné* [2000].