Additions to Part IV: Notes

I. Additions to note 120:

Replace 42. by:

42. $14 + 43 \leftrightarrow 407$, Bacsich [1972b].

and add:

64. $8 + 70 \leftrightarrow 8 + 406$, Alas [1994].
65. $8 + 385 \leftrightarrow 8 + 406$, Alas [1994].

II. Replace note 127 by the following:

NOTE 127

Forms [1 BZ] (Vector space multiple choice) and [1 DG] (Vector space Kinna-Wagner principle) were suggested by K. Keremedis. It is clear that [1 BZ] implies 346. In this note we prove that [1 DG] implies the Kinna-Wagner principle $KW(\infty, < \aleph_0)$ (form [62 E]). Since the axiom of choice is implied by the conjunction of forms 62 and 67, we obtain a proof that [1 DG] + 67 implies the axiom of choice. (Form 62 is $C(\infty, < \aleph_0)$ and 67 is the axiom of multiple choice.) Keremedis [1999d] proves that [1 DG] implies 67 to complete the proof that [1 DG] implies the axiom of choice. Similarly, since [1 BZ] implies form 67, we obtain the result: [1 BZ] implies the axiom of choice.

Let $X = \{y_i : i \in K\}$ be a family of finite sets. For each y_i let U_i be the real vector space $\mathbb{R}^{y_i}$ with pointwise addition and scalar multiplication. (If $y_i = \{a_1, \ldots, a_n\}$ we could think of U_i as being the set of all formal sums $k_1 a_1 + \cdots + k_n a_n$ where the k_i's are real.) Let S_i be the subspace $\{g \in U_i : g \text{ is constant}\}$. (Or in terms of formal sums, S_i is all formal sums $ka_1 + \cdots + ka_n$.) Let V_i be the quotient space $V_i = U_i/S_i$. That is, V_i consists of all equivalence classes $[g]$ of elements of U_i under the relation $g \sim f \Leftrightarrow g - f \in S_i$. By form [1 DG] there is a family $\{F_i : i \in K\}$ such that for each $i \in K$, F_i is an independent subset of V_i. Since U_i is finite dimensional F_i must be finite. Say $F_i = \{b_1, \ldots, b_r\}$ then since F_i is independent, the element $w_i = b_1 + \cdots + b_r$ is not zero. The vector $w_i = [g]$ for some $g \in U_i$. Assume $f \in [g]$. Then if $a, a' \in y_i$ and $g(a) \le g(a')$ it follows from the fact that $f - g$ is constant that $f(a) \le f(a')$. Therefore the set $K_i = \{a \in y_i : g(a) \text{ is minimum among the numbers } g(a') \text{ for } a' \in y_i\}$ is independent of the choice of $g \in w_i$. It is also true that $K_i \neq y_i$. This follows from the fact that $w_i \neq 0$ which implies that g is not constant. The family $\{K_i : i \in K\}$ is therefore a Kinna-Wagner function for X.

III. Add a new note 150.

NOTE 150

We give a proof that 385 (Every proper filter with a countable base over a set S (in $\mathcal{P}(S)$) can be extended to an ultrafilter.) implies 70 (There is a non-trivial ultrafilter on ω.) It is sufficient to show that there is a filter on ω with a countable base. For each $n \in \omega$ let $x_n = \{m \in \omega : m \ge n\}$. Then, $X = \{x_n : n \in \omega\}$ is a countable set which is a base for a filter $\mathcal{F}$ on ω.

IV. Add the following at the end of note 23:

(A *topological vector space (linear topological space)* is a vector space (linear space) with a topology in which the operations of addition and scalar multiplication are continuous. An *affine subspace*, A, of a topological vector space E is a translation of a subspace of E, $A = v + S = \{v + w : w \in S\}$, where S is a subspace of E and $v \in E$.)

V. Add the following right after the definition of locally convex in note 96:

(A subset S of X is *convex* if for all x and y in S, $L(x, y) \subseteq S$, where $L(x, y) = \{\lambda_1 x + \lambda_2 y : \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 + \lambda_2 = 1, \text{ and, } 0 \leq \lambda_1, \lambda_2 \leq 1\}$.