NOTE 157

A proof that form 133 implies form 340. (This proof is due to Eric Hall.) Form 133 is the statement: Every set that cannot be well ordered has an amorphous subset. (This is inf 2 in Rubin/Rubin [1985] where it is shown that inf 2 is equivalent to AC in ZF, but not in ZF^0.) Form 133 is true in $\mathcal{N}1$, $\mathcal{N}24$, $\mathcal{N}24(n)$, and $\mathcal{N}26$. Form 340 is: Every Lindelöf metric space is separable. We shall show that 133 implies 340.

proof. Suppose X with the metric d is a Lindelöf metric space. We shall prove, assuming form 133, that X is separable. First we shall show:

Lemma 1. *If M is an amorphous subset of X, then the range of $d/M \times M$ $(= d[M \times M])$, is finite.*

Proof. For each $m \in M$, $d[m \times M]$ is finite because M is amorphous. Thus,

$$d[M \times M] = \bigcup_{m \in M} d[m \times M]$$

is a finite union of finite sets, so it is finite

Lemma 2. *If X has an amorphous subset then X is not Lindelöf.*

Proof. Let M be an amorphous subset of X and let $S = d(a,b) : a,b \in M$. By Lemma 1, S is a finite set and it is clearly non-empty because M is infinite. Choose a positive ϵ which is less than $\frac{1}{2}$ of the minimum number in S. It follows from Lemma 1 that ϵ exists.

Let $B(\epsilon, m)$, where $m \in M$, be an open neighborhood about m with radius ϵ. Then

$$B(\epsilon, m) : m \in M \cup X \setminus \overline{M}$$

is an infinite open cover with no countable subcover, Thus, X is not Lindelöf.

It follows from Lemma 2 and 133 that X can be well ordered. Thus, since X is a Lindelöf metric space, it follows from form [8 V] that X is separable. $\square$