

In this note we summarize what is known about the relationships between possible definitions of “finite”.

Several authors have studied this topic they include Diel [1974], Howard/Yorke [1989], Levy [1958], Spišiak [1993], Spišiak/Vojtáš [1988], Tarski [1938b] and [1954a], Truss [1974a] and Hickman [1972]. The first eight definitions and their names are due to Tarski [1954a]

Definition.. A set A is said to be

decomposable if there are two pairwise disjoint non-empty sets x and y such that $x \prec A$, $y \prec A$, and $A = x \cup y$. (Similarly, a cardinal number m is said to be *decomposable*, if there exist two non-zero cardinal number p and q such that $p < m$, $q < m$, and $m = p + q$.)

Dedekind finite if it is not equivalent to a proper subset of itself. Otherwise, A is *Dedekind infinite*.

T-finite if every non-empty, monotone $X \subseteq \mathcal{P}(A)$ has a \subseteq -maximal element.

I-finite if every non-void family of subsets of A has an \subseteq -maximal element. (This is (equivalent to) the usual definition of finite.)

Ia-finite if it is not the union of two disjoint sets neither of which is I-finite.

II-finite if every non-empty \subseteq -monotone family of subsets of A has a \subseteq -maximal element.

III-finite if the power set of A is Dedekind finite.

IV-finite if A is Dedekind finite.

V-finite if $|A| = 0$ or $2 \cdot |A| > |A|$.

VI-finite if $|A| = 0$ or $|A| = 1$ or $|A|^2 > |A|$.

VII-finite if A is I-finite or not well orderable.

In Tarski [1954a] and Levy [1958] it is shown that if a set is finite according to any definition in the above list then it is finite according to any following definition and none of the reverse implications hold.

In Tarski [1938b] the following definitions are considered:

Definition. For each $n \in \omega$, A set A is $T(n)$ -finite if for every $S \subseteq \mathcal{P}(A)$, if every collection of n non-empty, pairwise disjoint subsets of A contains an element of S , then there are $2n + 1$ elements $x(i)$, $1 \leq i \leq 2n + 1$ of S such that $A \subseteq \bigcup \{x(i) : 1 \leq i \leq 2n + 1\}$.

In Howard/Yorke [1989] the following definition is introduced:

Definition. A set A is D finite if $|A| = 0$ or $|A| = 1$ or $A = C \cup D$ with $|C| < |A|$ and $|D| < |A|$. (Note that A is D finite if and only if $|A| = 0$ or $|A| = 1$ or A is decomposable.

It is shown there that $IV \rightarrow D$ and $D \rightarrow VII$. Further, $V \not\rightarrow D$ and $D \not\rightarrow VI$.

In Truss [1974a] the following classes of Dedekind finite cardinals are considered:

Definition.

$\Delta_1 = \{x : x = y + z \rightarrow y \text{ or } z \text{ is finite}\}$. (We will say A is Δ_1 finite if $|A| \in \Delta_1$. Δ_1 -finite is equivalent to Ia-finite.)

$\Delta_2 = \{|A| : \text{any linearly ordered partition of } A \text{ is finite}\}$. (Δ_2 -finite is equivalent to II-finite.)

$\Delta_3 = \{|A| : \text{any linearly ordered subset of } A \text{ is finite}\}$.

$\Delta_4 = \{x : \neg(\aleph_0 \leq^* x)\}$. (Δ_4 -finite is equivalent to III-finite.)

$\Delta_5 = \{x : \neg(x + 1 \leq^* x)\}$.

It is shown that $I\text{-finite} \rightarrow \Delta_1 \rightarrow \Delta_2 \rightarrow \Delta_4 \rightarrow \Delta_5 \rightarrow IV\text{-finite}$ and $\Delta_2 \rightarrow \Delta_3 \rightarrow IV\text{-finite}$ and that none of the implications are reversible. It is also shown that various combinations of equality and inequality between these classes are consistent. For example, it is shown that

$$ZF + \omega = \Delta_1 = \Delta_2 = \Delta_4 \neq \Delta_5 \neq IV\text{-finite} + \Delta_2 = \Delta_3 \neq IV\text{-finite}$$

is consistent.

In Diel [1974] two definitions are finite are considered: “Almost finite” which is equivalent to III-finite and “strongly Dedekind finite” which is equivalent to Δ_5 -finite.

Finally in Howard/Spišiak [1994], Spišiak [1993] and Spišiak/Vojtáš [1988] the following forms are considered:

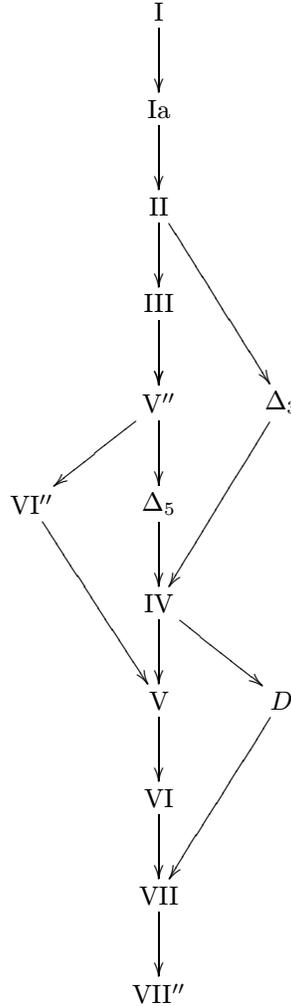
Definition. Let F be one of I, Ia, II, III, IV, V, VI or VII. Then a set A is F'' -finite if $\mathcal{P}(A)$ is F -finite.

It is shown that each of I'' , Ia'' , II'' and III'' are equivalent to I and that the following implications hold: $III \rightarrow V'' \rightarrow IV$, $V'' \rightarrow VI'' \rightarrow V$, and $VII \rightarrow VII''$. Further it is consistent (with ZF) that there exists a set with any combination of the properties III, V'' , VI'' , IV and V not excluded by the known implications described so far in this note. Finally, it is shown that form 88 ($C(\infty, 2)$) implies that V'' and III are equivalent.

In addition, in Howard/Yorke [1989], if J and K are two possible definitions of finite then the assertion that J and K are equivalent is denoted by $E(J, K)$ and the strength of these weak forms of AC is investigated. Several of these appear in the list of forms: $E(I, IV)$ is form 9, $E(I, Ia)$ is form 64, $E(I, III)$ is form 82, $E(I, II)$ is form 83, $E(II, III)$ is form 84, $E(II, IV)$ is form [9 B], $E(IV, V)$ is form [3 E], $E(V, VI)$ is form [1 AH], $E(VI, VII)$ is form [1 AI], $E(V'', III)$ is form 276 and $E(D, VII)$ is form 277.

We note that $E(I, D)$ which is equivalent to “every infinite set is decomposable” is equivalent to the axiom of choice since it is form CN 4F in H. Rubin/J. Rubin [1985], p. 139.

The relationships between the various definitions of finite are summarized in the following diagram:



(In addition $VI'' \not\rightarrow IV$, $IV \not\rightarrow VI''$, $V \not\rightarrow D$, $D \not\rightarrow VI$, and none of the arrows are reversible.)

Also, it was shown by De la Cruz [2001] that Δ_3 -finite and III-finite are independent, neither implies the other in ZF. Also, he has shown that $V'' \implies \Delta_5$. In $\mathcal{N}3$, Mostowski’s linearly ordered model, $\mathcal{P}(A)$, where

A is the set of atoms, is Dedekind finite, but can be linearly ordered. Therefore, $\mathcal{P}(A)$ is III-finite, but not Δ_3 -finite.

On the other hand, in Fraenkel's Basic Model, $\mathcal{N}1$, the set of all finite subsets of A , is III-infinite, but it contains no infinite linearly ordered subset. For suppose that L is such an infinite linearly ordered subset and E is a support for L and its linear order R . Then, there exists $n \in \omega$ such that $L' = \{x \in L : |x| = n\}$ is infinite; otherwise we can well order L by

$$x < y \text{ iff } |x| < |y| \text{ or } (|x| = |y| \text{ and } xRy).$$

Then we can define a map from A onto ω , which is impossible.

Now, since there are only finitely many subsets of E , there exist distinct $u, v \in L'$ such that

$$u \cap E = v \cap E.$$

It is easy to find a permutation σ that fixes E and such that $\sigma(u) = v$, $\sigma(v) = u$. This contradicts the assumption that R is a linear order supported by E .

It is also easy to get a permutation model which is half Mostowski and half basic Fraenkel, in which there exist sets which are Δ_3 -finite and not III-finite and other sets which are III-finite, but not Δ_3 finite.

Finally, the sentence:

" $(\exists A, B)$ (A is III-finite and A is not Δ_3 -finite, and B is Δ_3 -finite and B is not III-finite)"

is boundable, therefore the result transfers to ZF.

It follows from De la Cruz [2001] and the above comments, that none of the arrows in the above diagram are reversible in ZF, and if there is no arrow between two forms, then neither implies the other in ZF.